1 Introduction

2 Warm Up

1. 41: prime
2. 73: prime
3. 91: $91 = 7 \times 13$
4. 540: $540 = 2^2 \times 3^3 \times 5$
5. 5040: $5040 = 2^4 \times 3^2 \times 5 \times 7$

3 Infinitely Many Primes?

Here are all the prime numbers less than 100:
2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

1. Find the gaps between the consecutive prime numbers given above. (List them below)

   The gaps between consecutive prime numbers are: $1, 2, 2, 4, 2, 4, 2, 4, 6, 2, 6, 4, 2, 4, 6, 6, 2, 6, 4, 2, 6, 4, 6, 9$

2. As the prime numbers get larger, what gradually happens to the size of the gaps you found above?

   As the prime numbers get larger, the gaps between consecutive prime numbers become larger as well.

3. As we continue going to larger numbers, do you think we can keep finding more prime numbers?

   As we move to larger numbers, we should be able to keep finding prime numbers. This is because even though the gaps between consecutive primes become larger, the gaps are still finite. This means that there may be infinitely many prime numbers.

4 Twin Primes

1. Find all the *twin primes* among prime numbers less than 100.

   $(3, 5)$, $(5, 7)$, $(11, 13)$, $(17, 19)$, $(29, 31)$, $(41, 43)$, $(59, 61)$, $(71, 73)$

2. A prime number is called an *isolated prime* if it isn't part of a twin pair. Find all the isolated primes among prime numbers less than 100.

   The isolated primes less than 100 are: 2, 23, 37, 47, 53, 67, 79, 83, 89, 97

3. Euclid was also one of the first people to conjecture that there are an infinite number of twin primes. However, to this day, there is no proof to this yet. Do you think that there are infinitely many twin primes? (No wrong answers!)

   Answers can vary.

5 Proof by Contradiction

1. A number, A, is divisible by all prime numbers. Write an expression for A in terms of $p_1, p_2, ..., p_n$.

   $A = p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot ... \cdot p_n$

2. Write down an expression for $B = A + 1$ in terms of $p_1, p_2, ..., p_n$.

   $B = p_1 \cdot p_2 \cdot p_3 \cdot p_4 \cdot ... \cdot p_n + 1$

3. Is B divisible by any of the prime numbers $p_1, p_2, p_3, ..., p_n$? (Hint: Find the remainder when you divide B by each of the given prime numbers.)

   B is not divisible by any of the given prime numbers. If you divide B by p_1, the quotient is $p_2 \cdot p_3 \cdot p_4 \cdot ... \cdot p_n + \frac{1}{p_1}$. The quotient is not a natural number, and the remainder is not 0. Similarly with the other given prime numbers.

4. Using your answer above, can we conclude that B is prime? Why or why not?

   Since B is not divisible by any of the given prime numbers, it is also a prime number.

5. Why does this mean that we got a *contradiction* with our assumption?

   Since B is also a prime number and is larger than p_n, our assumption of a given set of prime numbers p_1, p_2, p_3, ..., p_n is incorrect.

6. What is your conclusion? Are there infinitely many primes?

   This must mean that the set of prime numbers is not finite and that there are infinitely many primes.